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PARTITION CONDITIONS AND VERTEX-CONNECTIVITY OF GRAPHS

by

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It was proved ([5], [6]) that if G is an n-vertex-connected graph then for any vertex sequence $v_1, \dots, v_n \in V(G)$ and for any sequence of positive integers $k_1, \dots, k_n$ such that $k_1 + \dots + k_n = |V(G)|$, there exists an n-partition of $V(G)$ such that this partition separates the vertices $v_1, \dots, v_n$, and the class of the partition containing v_i induces a connected subgraph consisting of k_i vertices, for $i = 1, 2, \dots, n$. Now fix the integers $k_1, \dots, k_n$. In this paper we study what can we say about the vertex-connectivity of G if there exists such a partition of $V(G)$ for any sequence of vertices $v_1, \dots, v_n \in V(G)$. We find some interesting cases when the existence of such partitions implies the n-vertex-connectivity of G, in the other cases we give sharp lower bounds for the vertex-connectivity of G.

1. Introduction

L. Lovász [6] and the present author [5] proved that if G is an n-vertex-connected graph then for every n vertices there exists an n-partition of the vertex-set of G such that this partition separates these n vertices, the classes of the partition contain given numbers of vertices and induce connected subgraphs of G.

By the definition of n-vertex-connectivity, if all but one of the classes are singletons, then the above-mentioned partition condition is both necessary and sufficient for G to be n-vertex-connected.

In this paper, we find some interesting cases when such conditions are sufficient for G to be n-vertex-connected. If these partition conditions are insufficient then we give sharp lower bounds for the vertex-connectivity of G and we find additional conditions which, together with the partition condition, will be sufficient for G to be n-vertex-connected.

2. Preliminaries

Throughout this paper, graph means finite undirected graph without loops and multiple edges. In general, we follow the terminology of Ore [11]. For completeness, we recall some definitions. $V(G)$ and $E(G)$ denotes the vertex-set and the edge-set of the graph G, further $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For $X \subseteq V(G)$, the induced

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subgraph $G(X)$ has vertex-set X and $e\in E(G(X))$ if and only if $e\in E(G)$ and its endpoints belong to X. A path is a sequence of distinct edges such that each pair of consecutive edges share a common vertex and no vertex appears more than once. For disjoint $X, Y\subseteq V(G)$, a path is an XY-path if one endpoint of it belongs to X the other to Y and no further vertex of the path belongs to $X\cup Y$.

An n-family of XY-paths is said to be *openly disjoint* if the paths are pairwise disjoint with respect to the vertices and the edges, except only for the common endpoints of the paths. For a graph G let $v(G)\geqq q+m$ and let $l_1, l_2, \dots, l_q$ and $k_1, k_2, \dots, k_m$ be two nonincreasing sequences of positive integers such that $\sum_{i=1}^{q} l_i=\sum_{j=1}^{m} k_j=n$. The graph G is said to have *property* $W_n(l_1, \dots, l_q; k_1, \dots, k_m)$ if for any $q+m$ distinct vertices $a_1, a_2, \dots, a_q, b_1, b_2, \dots, b_m\in V(G)$ there exists an openly disjoint n-family of AB-paths, where $A=\{a_1, \dots, a_q\}$, $B=\{b_1, \dots, b_m\}$ such that a_i and b_j are the endpoints of l_i and k_j of these paths, respectively.

A graph G is n-vertex-connected if $v(G)\geqq n+1$ and $G(V(G)-C)$ is connected for every $C\subseteq V(G)$ such that $|C|\leqq n-1$. The *vertex connectivity (number)* $\varkappa(G)$ is the maximum value of n for which G is n-vertex-connected.

Let G be a graph such that $v(G)\geqq n+1$ and let $v_1, v_2, \dots, v_n\in V(G)$ be distinct vertices. Let $k_1, k_2, \dots, k_n$ be a nondecreasing sequence of positive integers such that $\sum_{i=1}^{n} k_i=v(G)$. The graph G is said to satisfy the *partition condition*

$$P_n(v_1, v_2, \dots, v_n;\ k_1, k_2, \dots, k_n)$$

if there is a partition $\{V_1, V_2, \dots, V_n\}$ of $V(G)$ such that $v_i\in V_i$, $|V_i|=k_i$ and $G(V_i)$ is connected for $i=1, 2, \dots, n$. The graph G is said to satisfy the partition condition $P_n(k_1, k_2, \dots, k_n)$ if G satisfies the partition condition $P_n(v_1, v_2, \dots, v_n; k_1, k_2, \dots, k_n)$ for every choice of $v_1, v_2, \dots, v_n\in V(G)$. For the nondecreasing sequence $k_1, k_2, \dots, k_n$ of positive integers, let α_t denote the number of the indices $1\leqq i\leqq n$ such that $k_i=t$. Then, obviously,

$$\sum_{j=1}^{v(G)} \alpha_j = n, \quad \sum_{j=1}^{v(G)} j\alpha_j = v(G).$$

For a connected graph G, a subset $C\subseteq V(G)$ is called a *cutset* if the induced subgraph $G(V(G)-C)$ is not connected. A cutset having cardinality $\varkappa(G)$ is a minimum cutset. The set of all minimum cutsets of G is denoted by $C(G)$. Note that $C(G)=\emptyset$ if and only if G is a complete graph. For $C\in C(G)$, a component P of $G(V(G)-C)$ is called a *part* of G with respect to C. For $C\in C(G)$, $\{V_1^C, V_2^C\}$ is called a partition of G with respect to C if $V_1^C\cap V_2^C=\emptyset$, $V_1^C\cup V_2^C=V(G)-C$, and each induced subgraph $G(V_i^C)$, $(i=1, 2)$, is a union of a set of parts of G with respect to C. Without loss of generality, we assume that $|V_1^C|\leqq|V_2^C|$. (Clearly, C need not determine $\{V_1^C, V_2^C\}$ uniquely.) Further we define

$$p(G):=\min\{\min\{|V(P)|: P \text{ is a part of } G \text{ with respect to } C\}: C\in C(G)\}.$$

Note that $p(G)$ is not defined if and only if G is a complete graph. In the terms of the above notations, a graph G is n-vertex-connected if and only if G satisfies the partition condition $P_n(1, 1, \dots, 1, v(G)-n+1)$. Menger's theorem reads as follows:

Theorem 2.1. ([8], [12]) *A graph G is n-vertex-connected if and only if G satisfies* $W_n(n; n)$.

Dirac [1] proved that an n-vertex-connected graph G satisfies every condition of type W_n.

Theorem 2.2. ([1]) *Let G be a graph and let q, m, n be positive integers such that* $\varkappa(G) \geqq n$ *and* $q+m \leqq v(G)$. *Then G satisfies* $W_n(l_1, \dots, l_q; k_1, \dots, k_m)$ *for arbitrary positive integers* $l_1 \geqq l_2 \geqq \dots \geqq l_q$; $k_1 \geqq k_2 \geqq \dots \geqq k_m$ *such that* $\sum_{i=1}^{q} l_i = \sum_{j=1}^{m} k_j = n$.

Mesner, Watkins and Green studied the connection between general conditions of type W_n and the connectivity number $\varkappa(G)$. As an illustration we mention two theorems.

Theorem 2.3. ([9], [4]) *Let* $k_1, k_2, \dots, k_m$ *be a nonincreasing sequence of positive integers such that* $\sum_{j=1}^{m} k_j = n$. *If either* $v(G) \geqq 2n-2$ *or* $k_3 = k_4 = \dots = k_m = 1$ *then a necessary and sufficient condition that G satisfy* $W_n(n; k_1, \dots, k_m)$ *is that G be n-vertex-connected.*

Theorem 2.4. ([4]) *Let* $q \leqq n$ *and* $m \leqq n$ *be positive integers such that* $v(G) \geqq q+m$. *A necessary and sufficient condition that G be n-vertex-connected is that G satisfy* $W_n(n-q+1, 1, 1, \dots, 1; n-m+1, 1, 1, \dots, 1)$.

Further, Green [4] gave a sharp lower bound of $\varkappa(G)$ if G satisfies some $W_n(l_1, \dots, l_q; k_1, \dots, k_m)$ for arbitrarily given nonincreasing sequences $l_1, \dots, l_q$ and $k_1, \dots, k_m$ of positive integers such that $\sum_{i=1}^{q} l_i = \sum_{j=1}^{m} k_j = n$.

In this paper, we study the connection of $\varkappa(G)$ and the generalizations $P_n(k_1, \dots, k_n)$ of the defining property $P_n(1, 1, \dots, 1, v(G)-n+1)$. L. Lovász and the present author independently proved the following theorem which was conjectured by A. Frank [2] and a weaker version of which was conjectured by S. B. Maurer [7]. The two proofs are very different, Lovász used topological methods, the present author used only concepts of graph theory. Some special cases were proved previously by A. Frank [3] and K. Milliken [10]. This theorem corresponds to Dirac's theorem.

Theorem 2.5. ([5], [6]) *A graph G with* $v(G) \geqq n+1$ *is n-vertex-connected if and only if G satisfies the partition condition* $P_n(k_1, \dots, k_n)$ *for every nondecreasing sequence* $k_1, \dots, k_n$ *of positive integers such that* $\sum_{i=1}^{n} k_i = v(G)$.

The purpose of this paper is to study partition conditions $P_n(k_1, k_2, \dots, k_n)$ in analogy with the investigations of W_n by Mesner, Watkins and Green.

3. Necessary and sufficient partition conditions for n-vertex-connectivity

According to Theorem 2.5, any partition condition $P_n(k_1, \dots, k_n)$, $\left(\sum_{i=1}^{n} k_i = v(G)\right)$, is necessary for G to be n-vertex-connected. In the following theorems, we find some particular partition conditions $P_n(k_1, k_2, \dots, k_n)$ which are sufficient for G to be n-vertex-connected. First we derive a useful lower bound for $|V_1^C \cup C|$ when G satisfies $P_n(k_1, \dots, k_n)$.

Lemma 3.1. *Let G be a graph satisfying $P_n(k_1, \dots, k_n)$ and let $v(G) = \sum_{i=1}^{n} k_i \geqq n+1$. Let $C \in C(G)$. Then every partition $\{V_1^C, V_2^C\}$ of G with respect to C satisfies the inequality*

$$|V_2^C \cup C| \geqq |V_1^C \cup C| \geqq n+1.$$

Proof. Suppose that $|V_1^C \cup C| \leqq n$. Choose $v_1, v_2, \dots, v_n$ such that $V_1^C \cup C \subseteqq \subseteqq \{v_1, v_2, \dots, v_n\}$, $v_n \in V_1^C$. We have $k_n \geqq 2$ because of $v(G) \geqq n+1$, but every neighbour of v_n is an element of $\{v_1, \dots, v_{n-1}\}$ and so G does not satisfy $P_n(v_1, \dots, v_n; k_1, \dots, k_n)$, contradiction. ∎

Theorem 3.2. *Let G be a graph such that $v(G) \geqq n+1$ and let $k_1, k_2, \dots, k_n$ be a non-decreasing sequence of positive integers such that $\sum_{i=1}^{n} k_i = v(G)$, $\alpha_1 + \alpha_2 \geqq n-1$. Then G is n-vertex-connected if and only if it satisfies the partition condition $P_n(k_1, k_2, \dots, k_n)$.*

Proof. If G is n-vertex-connected then it satisfies $P_n(k_1, \dots, k_n)$ by Theorem 2.5. Let G satisfy $P_n(k_1, \dots, k_n)$. If G is a complete graph then $\varkappa(G) \geqq n$ because of $v(G) \geqq \geqq n+1$. So we may assume that G is not a complete graph, $C(G) \neq \emptyset$. Let $C \in C(G)$ and let $\{V_1^C, V_2^C\}$ be a partition of G with respect to C. $|V_2^C \cup C| \geqq |V_1^C \cup C| \geqq n+1$ by Lemma 3.1. Suppose that $|C| = \varkappa(G) = r \leqq n-1$. We distinguish two cases.

Case 1. $|V_2^C| > \sum_{i=1}^{r} k_i - r$.

Then choose $v_1, v_2, \dots, v_n$ so that

$$C = \{v_1, v_2, \dots, v_r\}, \quad \{v_{r+1}, v_{r+2}, \dots, v_n\} \subset V_1^C.$$

Then there exists a partition $\{V_1, V_2, \dots, V_n\}$ of $V(G)$ such that $v_i \in V_i$, $|V_i| = k_i$, $G(V_i)$ is connected by $P_n(v_1, v_2, \dots, v_n; k_1, k_2, \dots, k_n)$. But then $\bigcup_{i=r+1}^{n} V_i$ has to be a subset of V_1^C obviously, and so

$$|V_2^C| \leqq \sum_{i=1}^{r} |V_i - C| = \sum_{i=1}^{r} (k_i - 1),$$

a contradiction.

Case 2. $|V_2^C| \leqq \sum_{i=1}^{r} k_i - r$.

Let $s = |V_1^C|$. Then $s \leqq |V_2^C| \leqq \sum_{i=1}^{r} k_i - r \leqq r < n$. Hence $\alpha_1 \leqq r - s < n - s$. Choose $v_1, v_2, \dots,$

$\ldots, v_n$ so that $V_1^C=\{v_{n-s+1}, \ldots, v_n\}$, $\{v_1, \ldots, v_{n-s}\}\subseteqq C$. (We can do it since $|V_1^C\cup C|>n$.) Now there exists a partition $\{V_1, \ldots, V_n\}$ of $V(G)$ such that $v_i\in V_i$, $|V_i|=k_i$, $G(V_i)$ is connected by $P_n(v_1, \ldots, v_n; k_1, \ldots, k_n)$, $(i=1, 2, \ldots, n)$. For $i=n-s+1, \ldots, n$, we have $k_i\geqq 2$ and so V_i has to contain an element of C. So $|C|\geqq n>r$, a contradiction. ∎

An interesting special case of the theorem is

Corollary 3.3. *Let G be a graph such that $v(G)=2n$. G is n-vertex-connected if and only if, for arbitrary vertices $v_1, \ldots, v_n\in V(G)$ there exists a perfect matching $\{e_1, \ldots, e_n\}\subseteqq E(G)$ such that e_i is incident to v_i $(i=1, \ldots, n)$.* ∎

The following simple theorem states that if $\varkappa(G)<n$ for a graph satisfying $P_n(k_1, \ldots, k_n)$ then the minimum cutsets do not cut the graph into very small pieces.

Theorem 3.4. *Let G be a graph such that $v(G)\geqq n+1$ and let $k_1, k_2 \ldots, k_n$ be a nondecreasing sequence of positive integers such that $\sum_{i=1}^{n} k_i=v(G)$, $k_n>p(G)$. Then $P_n(k_1, \ldots, k_n)$ is a necessary and sufficient condition for G to be n-vertex-connected.*

Proof. If G is n-vertex-connected then it satisfies the partition condition $P_n(k_1, \ldots, k_n)$ by Theorem 2.5.

Let G satisfy the inequality $p(G)<k_n$ and the partition condition $P_n(k_1, \ldots, k_n)$. Let $C\in C(G)$ be a minimum cutset and let $\{V_1^C, V_2^C\}$ be a partition of G with respect to C such that $|V_1^C|=p(G)$. Suppose that $|C|=\varkappa(G)<n$. Then choose $v_1, v_2, \ldots, v_n$ so that $C\subseteqq\{v_1, v_2, \ldots, v_{n-1}\}$, $v_n\in V_1^C$. Then there is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that $v_i\in V_i$, $|V_i|=k_i$ and $G(V_i)$ is connected for $i=1, 2, \ldots, n$ by $P_n(v_1, v_2, \ldots, v_n; k_1, k_2, \ldots, k_n)$. $V_n\cap C=\emptyset$ and $G(V_1)$ is connected, so $V_n\subseteqq V_1^C$. But $|V_n|=k_n>p(G)=|V_1^C|$, contradiction. ∎

Next we prove a simple lemma.

Lemma 3.5. *If G is a graph satisfying $P_n(k_1, \ldots, k_n)$ and $v(G)\geqq n+1$ then G is (at least) (α_1+1)-vertex-connected.*

Proof. If G is a complete graph then $\varkappa(G)\geqq n>\alpha_1$ because of $v(G)\geqq n+1$. So we may assume that G is not a complete graph, $C(G)\neq\emptyset$. Let $C\in C(G)$ and let $\{V_1^C, V_2^C\}$ be a partition of G with respect to C. We have $|V_1^C\cup C|\geqq n+1$ by Lemma 3.1. If $|C|=r\leqq\alpha_1$ then choose $v_1, \ldots, v_n$ so that $C=\{v_1, \ldots, v_r\}$ and $\{v_{r+1}, \ldots, v_n\}\subset V_1^C$. Then by $P_n(v_1, \ldots, v_n; k_1, \ldots, k_n)$, there exists a partition $\{V_1, \ldots, V_n\}$ of $V(G)$ such that $v_i\in V_i$, $|V_i|=k_i$ and $G(V_i)$ is connected for $i=1, \ldots, n$. But then $\bigcup_{i=1}^{n} V_i\subseteqq V_1^C\cup C$ by the connectivity of $G(V_i)$ $(i=1, \ldots, n)$. Then $V_2^C=\emptyset$, a contradiction. ∎

Using this lemma and Theorem 3.4, we give a further necessary and sufficient partition condition for a graph G to be n-vertex-connected. This condition does not depend on $p(G)$.

Theorem 3.6. *Let G be a graph such that $v(G)\geqq n+1$ and let $k_1, \ldots, k_n$ be a nondecreasing sequence of positive integers such that $\sum_{i=1}^{n} k_i=v(G)$, $\alpha_1\geqq n-2$. Then a necessary* *and sufficient condition for G to be n-vertex-connected is that G satisfies* $P_n(k_1, \ldots, k_n) = P_n(1, 1, \ldots, 1, k_{n-1}, k_n)$.

Proof. If G is n-vertex-connected then it satisfies $P_n(k_1, \ldots, k_n)$ by Theorem 2.5.

Let G satisfy $P_n(1, 1, \ldots, 1, k_{n-1}, k_n)$. If G is a complete graph then $\varkappa(G) \geqq n$ because of $v(G) \geqq n+1$. So we may assume that G is not a complete graph, $C(G) \neq \emptyset$. Let $C \in C(G)$ and let $\{V_1^C, V_2^C\}$ be a partition of G with respect to C. If $k_n > |V_1^C| \geqq p(G)$ then G is n-vertex-connected by Theorem 3.4. So we may assume that $k_n \leqq |V_1^C|$. Then $k_{n-1} \leqq k_n \leqq |V_1^C| \leqq |V_2^C|$ and so

$$v(G)-(n-2) = k_n + k_{n-1} \leqq |V_1^C| + |V_2^C| = v(G) - |C|.$$

Then $|C| \leqq n-2$, but this is impossible by Lemma 3.5. ∎

The following theorem treats the case when k_n is not small compared to $v(G)$. If $k_1, k_2, \ldots, k_{n-1}$ are fixed and $v(G)$ is large enough then n-vertex-connectivity is implied by $P_n\left(k_1, \ldots, k_{n-1}, v(G) - \sum_{i=1}^{n-1} k_i\right)$.

Theorem 3.7. *Let G be a graph such that* $v(G) \geqq n+1$ *and let* $k_1, k_2, \ldots, k_{n-1}$ *be a nondecreasing sequence of positive integers. If* $2k_n > v(G)-n+1$ *then a necessary and sufficient condition for G to be n-vertex-connected is that G satisfies* $P_n(k_1, \ldots, k_n)$, *where* $k_n := v(G) - \sum_{i=1}^{n-1} k_i$. ($v(G) - \sum_{i=1}^{n-1} k_i \geqq k_{n-1}$ *if* $v(G) > 2\sum_{i=1}^{n-1} k_i - n + 1$, *so* $k_1, \ldots, k_n$ *is non-decreasing.*)

Proof. If G is n-vertex-connected then it satisfies the partition condition $P_n\left(k_1, \ldots, k_{n-1}, v(G) - \sum_{i=1}^{n-1} k_i\right)$ by Theorem 2.5. Let G satisfy the inequality $2k_n > v(G)-n+1$ and the partition condition $P_n(k_1, \ldots, k_{n-1}, k_n)$. If G is a complete graph then it is n-vertex-connected by $v(G) \geqq n+1$. So we may assume that G is not complete, $C(G) \neq \emptyset$. Let $C \in C(G)$ and suppose that $|C| = r < n$. Let $\{V_1^C, V_2^C\}$ be a partition of G with respect to C. Then $|V_1^C \cup C| > n$ by Lemma 3.1. Choose $v_1, v_2, \ldots, v_n$ so that $\{v_1, \ldots, v_r\} = C$, $\{v_{r+1}, \ldots, v_n\} \subset V_1^C$.

Then there is a partition $\{V_1, \ldots, V_n\}$ of $V(G)$ such that $v_i \in V_i$, $|V_i| = k_i$ and $G(V_i)$ is connected ($i = 1, 2, \ldots, n$) by $P_n(v_1, \ldots, v_n; k_1, \ldots, k_n)$. $V_i \cap C = \emptyset$ if $i \geqq r+1$ and so

$$\bigcup_{i=r+1}^{n} V_i \subseteqq V_1^C, \quad \sum_{i=r+1}^{n} k_i \leqq |V_1^C|, \quad \sum_{i=1}^{r} (k_i - 1) \geqq |V_2^C| \geqq |V_1^C| \geqq \sum_{i=r+1}^{n} k_i.$$

So

$$v(G) = \sum_{i=1}^{n} k_i \leqq 2\sum_{i=1}^{r} k_i - r \leqq 2\sum_{i=1}^{n-1} k_i - (n-1),$$

a contradiction. ∎

4. Lower bounds of $\varkappa(G)$

Theorems 3.2. to 3.7. state necessary and sufficient conditions for a graph G to be n-vertex-connected. In this section, we extend these sufficiency arguments to provide lower bounds for vertex-connectivity. Sharpness of the lower bounds will be shown by examples.

Theorem 4.1. *Let G be an arbitrary incomplete graph such that $v(G)\geqq n+1$ and let $k_1, k_2, \ldots, k_n$ be a nondecreasing sequence of positive integers such that $\sum_{i=1}^{n} k_i = v(G)$. Let*

$$D_G := \min\left\{d > 0 \middle| \max_{\max(\alpha_1-d,1)\leqq i\leqq n-d}\left(\sum_{j=i+1}^{i+d} k_j+n-i-d\right) > p(G)\right\}.$$

If G satisfies the partition condition $P_n(k_1, \ldots, k_n)$, then

$$\varkappa(G) \geqq 1+n-D_G.$$

Remark. For any value of p, the following graphs G_0 satisfy $P_n(k_1, \ldots, k_n)$ and $\varkappa(G_0)=1+n-D_{G_0}$ (Fig. 1):

$$V(G_0) = A\cup B\cup C, \quad A\cap B = B\cap C = A\cap C = \emptyset,$$

$$|C| = 1+n-G_{G_0}, \quad p = p(G_0) = |A| \leqq |B| = v(G_0)-p(G_0)-|C|,$$

$$E(G_0) = \{(x, y\colon x, y\in V(G_0),\ x \neq y \text{ and } (x, y) \text{ is not an } AB\text{-path}\}.$$

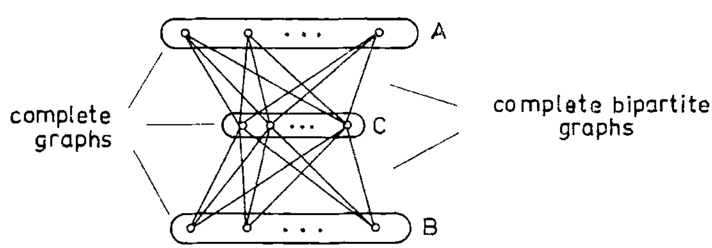

Fig. 1

(The proof of this is tedious but routine, so we omit it.)

Proof. Let G satisfy the partition condition $P_n(k_1, \ldots, k_n)$. Suppose that $\varkappa(G) \leqq n-D_G$ that is $|C|=r\leqq n-D_G$ for $C\in C(G)$. (G is incomplete, so $C(G)\neq\emptyset$.) Choose a minimum cutset $C\in C(G)$ and a partition $\{V_1^C, V_2^C\}$ of G with respect to C such that $|V_1^C|=p(G)$. We have $|V_1^C\cup C|>n$ by Lemma 3.1.

By the definition of D_G, there exists at least one index i satisfying $\max(\alpha_1-D_G, 1)\leqq i\leqq n-D_G$ such that $\sum_{j=i+1}^{i+D} k_j+n-i-D_G>p(G)$. Let i_0 be the greatest of these indices. We distinguish four cases.

Case 1. $\alpha_1>i_0$.

$\alpha_1 - D_G \leqq i_0$ and so $i_0 + D_G \geqq \alpha_1$. If moreover $i_0 + D_G < n$ then not only

$$\sum_{j=i_0+1}^{i_0+D_G} k_j + n - i_0 - D_G > p(G),$$

but

$$\sum_{j=i_0+2}^{i_0+1+D_G} k_j + n - (i_0+1) - D_G > p(G),$$

as well, since then the sum of k_j's increases at least by one ($k_{i_0+1}=1$, but $k_{i_0+1+D_G} \geqq 2$), and the subtracted i_0 increased only by one. So if $\alpha_1 > i_0$ then $i_0 + D_G = n$ by the maximality of i_0. Now, choose $v_1, v_2, \ldots, v_n$ so that $\{v_1, v_2, \ldots, v_r\} = C$, $\{v_{r+1}, \ldots, v_n\} \subset V_1^C$. Then there is a partition $\{V_1, \ldots, V_n\}$ satisfying $P_n(v_1, \ldots, v_n; k_1, \ldots, k_n)$ such that $v_i \in V_i$, $|V_i| = k_i$ and $G(V_i)$ is connected for $i = 1, \ldots, n$. But then $V_i \cap C = \emptyset$ and so $V_i \subseteqq V_1^C$ necessarily for $i \geqq r+1$, that is

$$|V_1^C| \geqq \sum_{j=r+1}^{n} k_j.$$

But then

$$v(G) = |V_2^C| + |C| + |V_1^C| \geqq |V_2^C| + r + \sum_{j=r+1}^{n} k_j \geqq$$

$$\geqq |V_2^C| + r + (n - D_G - r) + \sum_{j=n-D_G+1}^{n} k_j > |V_2^C| + r +$$

$$+ (n - D_G - r) + p(G) = v(G) + (n - D_G - r) \geqq v(G),$$

contradiction.

Case 2. $\alpha_1 \leqq i_0$ and $n - i_0 > |V_1^C| = p(G)$.

Now, choose $v_1, v_2, \ldots, v_n$ so that $\{v_1, v_2, \ldots, v_{n-p(G)}\} \subset C$ (we can do it by $|V_1^C \cup C| > n$), and $\{v_{n-p(G)+1}, \ldots, v_n\} = V_1^C$. But $n - p(G) + 1 > n - (n - i_0) + 1 > i_0 \geqq \alpha_1$ and so $k_i \geqq 2$ for $i \geqq n - p(G) + 1$. Then for the partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ satisfying $P_n(v_1, v_2, \ldots, v_n; k_1, k_2, \ldots, k_n)$, we have $V_i \cap C \neq \emptyset$ if $i \geqq n - p(G) + 1$ and so $|C| \geqq n - D_G$, contradiction.

Case 3. $\alpha_1 \leqq i_0$, $|V_1^C| = p(G) \geqq n - i_0$ and $r < i_0$.

Now, choose $v_1, v_2, \ldots, v_n$ so that $\{v_1, \ldots, v_r\} = C$, $\{v_{r+1}, \ldots, v_n\} \subseteqq V_1^C$. (We can do it by $|V_1^C \cup C| > n$.) But then for the partition $\{V_1, \ldots, V_n\}$ of $V(G)$ satisfying $P_n(v_1, \ldots, v_n; k_1, \ldots, k_n)$, $V_i \cap C = \emptyset$, $V_i \subseteqq V_1^C$ if $i \geqq r+1$ that is

$$\sum_{j=r+1}^{n} k_j \leqq |V_1^C|.$$

But then

$$v(G) = |V_1^C| + |C| + |V_2^C| \geqq \sum_{j=r+1}^{n} k_j + |C| + |V_2^C| \geqq (n - r - D_G) + \sum_{j=i_0+1}^{i_0+D_G} k_j +$$

$$+ |C| + |V_2^C| = (i_0 - r) + \left[(n - i_0 - D_G) + \sum_{j=i_0+1}^{i_0+D_G} k_j\right] + |C| + |V_2^C| >$$

$$> p(G) + |C| + |V_2^C| = v(G),$$

contradiction.

Case 4. $\alpha_1 \leqq i_0$, $|V_1^C| = p(G) \geqq n - i_0$ and $i_0 \leqq r$.

Now, choose $v_1, v_2, \ldots, v_n$ so that $\{v_1, v_2, \ldots, v_{i_0}\} \subseteqq C$, $\{v_{i_0+1}, \ldots, v_n\} \subseteqq V_1^C$. Then there exists a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that $v_i \in V_i$, $|V_i| = k_i$ and $G(V_i)$ is connected $(i=1, \ldots, n)$ by $P_n(v_1, v_2, \ldots, v_n;\ k_1, k_2, \ldots, k_n)$. But then at most $r - i_0$ sets from among $V_{i_0+1}, \ldots, V_n$ can contain an element of C. So at least $n - r$ of the sets $V_{i_0+1}, \ldots, V_n$ have to be placed disjointly in $V_1^C - \{v_1, \ldots, v_n\}$, that is the following inequality holds

$$p(G) = |V_1^C| \geqq \sum_{j=i_0+1}^{i_0+n-r} k_j + n - (i_0 + n - r).$$

But $\alpha_1 \leqq i_0$ and so

$$p(G) \geqq \sum_{j=i_0+1}^{i_0+n-r} k_j + n - (i_0+n-r) \geqq \sum_{j=i_0+1}^{i_0+D_G} k_j + 2(n-r-D_G) +$$

$$+ n - (i_0+n-r) = \sum_{j=i_0+1}^{i_0+D_G} k_j + 2n - r - 2D_G - i_0.$$

But then

$$v(G) = |V_1^C| + |C| + |V_2^C| \geqq \sum_{j=i_0+1}^{i_0+D_G} k_j + 2n - r - 2D_G - i_0 + |C| + |V_2^C|$$

$$= \left(\sum_{j=i_0+1}^{i_0+D_G} k_j + n - i_0 - D_G\right) + n - r - D_G + |C| + |V_2^C| > p(G) + |C| + |V_2^C| = v(G),$$

contradiction. And so we obtained contradiction in every case, that is

$$|C| = r > n - D_G, \quad \varkappa(G) \geqq 1 + n - D_G.$$

This completes the proof of Theorem 4.1. ▮

We can give also such an estimation for $\varkappa(G)$ that the bound does not depend on the structure of the graph G.

Theorem 4.2. *Let G be an arbitrary incomplete graph such that $v(G) \geqq n+1$ and let $k_1, \ldots, k_n$ be a nondecreasing sequence of positive integers such that $\sum_{i=1}^{n} k_i = v(G)$. Let*

$$D := \min\left\{d > 0:\ \max_{\max(\alpha_1 - d, 1) \leqq i \leqq n-d} \left[\sum_{j=i+1}^{i+d} k_j + n - i - d\right] > (v(G) - n + d)/2\right\}.$$

If G satisfies the partition condition $P_n(k_1, \ldots, k_n)$ then

$$\varkappa(G) \geqq 1 + n - D.$$

Remark. The following graph G_0 satisfies $P_n(k_1, \ldots, k_n)$ and $\varkappa(G_0) = 1 + n - D$:

$$V(G_0) = A \cup B \cup C, \quad A \cap B = B \cap C = A \cap C = \varnothing$$

$$|C| = 1 + n - D, \quad |A| = \left[\frac{v(G_0) - 1 - n + D}{2}\right], \quad |B| = \left\lceil\frac{v(G_0) - 1 - n + D}{2}\right\rceil,$$

$$E(G_0) = \{(x, y):\ x, y \in V(G_0),\ x \neq y \text{ and } (x, y) \text{ is not an } AB\text{-path}\}.$$

(See Fig. 1.) The proof is easy but tedious again.

Proof of Theorem 4.2. Although Theorem 4.2 is not an immediate consequence of Theorem 4.1, almost the same proof applies. We leave the details to the reader. ∎

Corollary 4.3. *Let G be an arbitrary graph such that $v(G)\geqq n+1$. Further let $k_1, \dots, k_n$ be a nondecreasing sequence of positive integers such that $\sum_{i=1}^{n} k_i = r(G)$. If G satisfies the partition condition $P_n(k_1, k_2, \dots, k_n)$ then*

$$\varkappa(G) \geqq \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Proof. For every positive nondecreasing sequence $k_1, k_2, \dots, k_n$

$$\sum_{i=\lfloor n/2\rfloor+1}^{n} k_i > \frac{v(G)-n+(n-\lfloor n/2\rfloor)}{2},$$

and so $D\leqq n-[n/2]$ in Theorem 4.2, hence $\varkappa(G)\geqq[n/2]+1$. ∎

Consider that the partition condition $P_n\big(1, 1, \dots, 1, v(G)-n+1\big)$ is the definition of n-vertex-connectedness, that if $\alpha_1+\alpha_2\geqq n-1$ or $\alpha_1\geqq n-2$ then

$$P_n\left(k_1, k_2, \dots, k_{n-1}, v(G)-\sum_{i=1}^{n-1} k_i\right)$$

guarantees n-vertex-connectedness of G and consider the ideas of the different arguments. We have the feeling that if $k_1\leqq k_1^*, \dots, k_{n-1}\leqq k_{n-1}^*$ and $\sum_{i=1}^{n} k_i = \sum_{i=1}^{n} k_i^*$ then $P_n(k_1, \dots, k_n)$ garantees a better lower bound for $\varkappa(G)$ than $P_n(k_1^*, \dots, k_n^*)$. And really, this happens to the bounds in Theorem 4.2 or Theorem 4.1. Let

$$f_m(k_1, \dots, k_{n-1}) =$$
$$= \min\left\{\varkappa(G)\colon\ v(G) = m,\ G \text{ satisfies } P_n(k_1, \dots, k_n) \text{ for } k_n = m-\sum_{i=1}^{n-1} k_i\right\}$$

for any nondecreasing sequence of positive integers $k_1, \dots, k_{n-1}$ such that $m-\sum_{i=1}^{n-1} k_i \geqq k_{n-1}$. Then the function f_m is monotonic nonincreasing. It would be nice to know if not only f_m but the truth value of P_n is a monotonic function of $k_1, \dots, k_{n-1}$ in the following sense:

Conjecture. Let G be an arbitrary graph such that $v(G)\geqq n+1$. Further let $k_1, \dots, k_n$ and $k_1^*, \dots, k_n^*$ be nondecreasing sequences of positive integers such that $k_1 \leqq k_1^*, \dots, k_{n-1}\leqq k_{n-1}^*$ and $\sum_{i=1}^{n} k_i = \sum_{i=1}^{n} k_i^* = v(G)$. If G satisfies the partition condition $P_n(k_1, \dots, k_n)$ then G satisfies the partition condition $P_n(k_1^*, \dots, k_n^*)$ as well.

This conjecture generalizes Theorem 2.5.

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